# Complex Variables and Conformal Mapping Applications in Fluid Flow Dynamics

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## Abstract

Transformations in complex analysis can lead to geometric representations of vector fields on the complex plane. These interpretations can be quite helpful in the analysis of functions of complex variables. Many of the functions treated will be applications of harmonic functions. Specific topics addressed in the presentation will include two-dimensional fluid flows, Pólya-Latta vector fields, and the Joukowski airfoil. In addition to the main focus of hydrodynamics and fluid flow, interesting results in computer graphics and hydrodynamics will also be presented.


## Introduction

Velocity Vector (vector field)

$V(x,y) = p(x,y) + iq(x,y)$

Equation of Continuity

[illegible]

Complex Potential

[illegible]

Equipotentials and Streamlines describe the paths of fluid particles ($\kappa_1$, $\kappa_2$ vary within an interval)

Equipotentials (velocity potential) $\phi(x,y) = \kappa_1$
Streamlines (stream function) $\psi(x,y) = \kappa_2$

Invariance of Flow Theorem

Let [illegible]

denote the complex potential for a fluid flow in a domain G in the w plane, where the velocity is

[illegible]

If the function

[illegible]

is a one-to-one conformal mapping from a domain D in the z plane onto F, then the composite function

[illegible]

is the complex potential for a fluid flow in D, where the velocity is

[illegible]

## Applications of Two-Dimensional Fluid Flow Mappings

- Joukowski airfoil
  - The image of a circle passing through $z=1$ and containing the point $z=-1$ is mapped onto a curve shaped like the cross section of an airplane wing
- Pólya-Latta vector fields
  - About polya-latta vector fields
  - Pólya plots can be created in *Mathematica* using **PlotPolyaField[f,{x,min,max},{y,min,max}]**

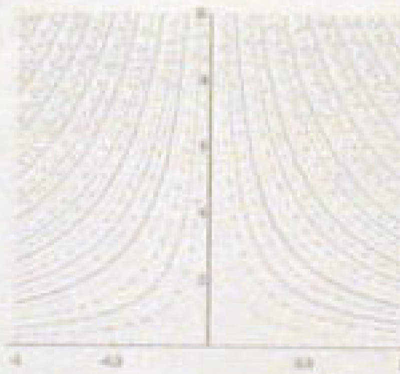
## Fluid Flow around a Wall



$$f[k\_, x\_] := k\sqrt{\frac{1+k^2+x^2}{k^2+x^2}}$$

The figure below depicts the complex potential for an ideal fluid flowing from left to right, parallel to $x$, across the complex plane. A wall stretches from from $-i$ to $i$. Using conformal mapping, $S(z)=(z+i)^{1/2}(z-i)^{1/2}$, streamlines can be parameterized and solved to obtain the above equation. Thus, points can be plotted as $k$ is varied for each streamline in the domain.

## Flow Towards a Wall



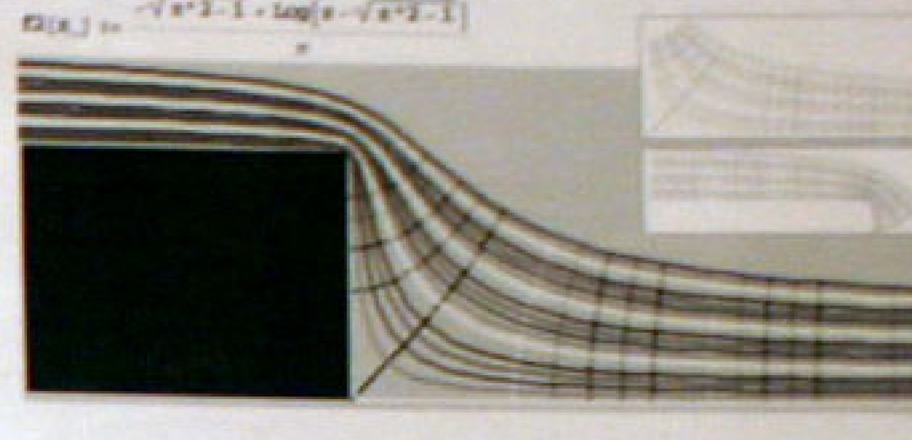
$$f[k\_, x\_] := \frac{k}{4x}$$

The complex potential $F(z)=A/2*z^2$ is depicted below, where $A=4$, and is always positive, in this case. The velocity potential, $\varphi(x,y)=A/2*(x^2-y^2)$, and stream function, $\psi(x,y)=Axy$, forms a family of hyperbolas with asymptotes along both axes. The fluid flows downwards along the streamlines as if parallel to the $iy$ axis and spreads out against a wall, the $x$ axis.

## Piecewise Representation of Flow over an Obstacle

$$f1[x\_] := \frac{\sqrt{x^2-1} + Log[x + \sqrt{x^2-1}]}{\pi},$$

$$f2[x\_] := \frac{-\sqrt{x^2-1} + Log[x - \sqrt{x^2-1}]}{\pi}$$



## Complex Mapping with Built-in Functions in *Mathematica*

The left plot uses a spherical Bessel function in **PolarPlot**.

A fluid hugs the walls of a box in the plot above. The argument for **CartesianPlot** is $f(z)=arcsin(z)$.

## Conclusions

In using the concept of an ideal fluid flow across the complex plane, we first defined the velocity vector, followed by the complex potential. Only fluids of irrotational flow were considered. Using the equation of continuity, equipotentials and streamlines were defined. These functions can easily be plotted using *Mathematica*. The visualization of such fluid flows is important for understanding ideal effects, such as those observed with the Joukowski Airfoil. Conformal mapping techniques can also aid in visualizing functions of complex variables.

## Resources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