A RESOURCE UNIT FOR ELEMENTARY ALGEBRA

BY

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A RESOURCE UNIT FOR ELEMENTARY ALGEBRA

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CHAPTER I

INTRODUCTION

Algebra has acquired the widespread reputation of being one of the most troublesome and difficult courses in the secondary school program. It has achieved this reputation because of inherent differences between algebra and any study which the student will have encountered earlier in his educational career. A large share of the difficulties in the study of algebra can be traced to the fact that the subject presents a new and different approach to the study of quantitative relationships, characterized by a new symbolism, new concepts, a new language, and a much higher degree of generalization and abstraction than has been previously encountered.

Statement of the Problem

The purpose of this study is to prepare a resource unit for elementary algebra which will provide the teacher with a variety of materials and suggestions for more effective teaching. As a supplement to the basic material of the textbook, the resource unit will provide illustrations, activities, teaching aids, and evaluative procedures for building a learning unit. Practical application of the principles learned will also be included.

The unit will deal with three major topics covered during the first semester course in algebra. The topics to be included are literal numbers and formulas, positive and negative numbers, and equations.
Characteristics of a Resource Unit

The resource unit has been defined as:

A systematic and comprehensive survey, analysis, and organization of the possible resources (e.g. problems, issues, activities, bibliographies, etc.) which a teacher might utilize in planning, developing, and evaluating a learning unit.¹

Resource units can be characterized by such terms as (1) comprehensiveness: several learning units can evolve from one source unit; (2) flexibility: parts of the resource unit can be employed to help the teacher handle individual differences; (3) correlations: there is a cutting across of subject matter lines to establish relationships between subjects; (4) community resources: the community is studied to ascertain its possible contribution to the objectives of the unit; and (5) survey and analysis: major problems and critical issues are treated in an objective manner, presenting all sides of debatable subjects. Resource unit suggestions are consistent with today's research in child growth and development and with current studies in the field of learning.²

The main purpose for which resource units are developed is to help teachers prepare for the process of planning learning experiences with their students. Resource units are for the use of the teachers and are not intended to be placed in the hands of students. They are designed to serve and to help teachers and should be evaluated on that basis. Most resource units include many more suggestions than can be incorporated in any unit developed by teachers and their classes. They are designed to promote flexibility and democratic group planning in the curriculum.

One of the best tests to find weaknesses in the unit is to try it in actual teaching. As time goes on the necessity for revision will be apparent. Only through constant trial and revision can a unit be of


real service to teachers and students. As the unit is used and improved, teachers will undoubtedly be able to point out many other problems which should have been solved as the unit was developed.

Procedure Followed

Before attempting to write this unit it was necessary for the writer to investigate the field of resource units to determine what had been done on this subject in the past. This included contacting the Iowa State Department of Education, the National Education Association Research Department, the United States Office of Education, and several staff members of the Department of Education at Drake University. Textbooks on curriculum development, periodicals, and research bulletins were also checked. With a limited number of resource units available on this subject, all information pertaining to the selected topic of algebra was collected from periodicals, textbooks, research bulletins, yearbooks, and film services. Reference material was used to formulate the specific objectives of each topic and to list activities that would enrich the teaching program of the topic.

Organization of the Report

The material for the unit is divided into three separate topics. The first topic deals with literal numbers and formulas; the second, with equations; and the third, with positive and negative numbers. A general outline was developed for each topic and includes:

1. Background expected of students with reference to the topic.

2. Particular understandings and abilities which the student should acquire or strengthen through his study of this topic.
3. Activities, procedures, and resources that will enable the student to gain these desired understandings and abilities most effectively.

4. Special difficulties students may be expected to encounter in acquiring the desired understandings and skills.

5. Suggestions, devices, and procedures most likely to help the student overcome these difficulties and to avoid these mistakes.

Also included within the general outline is the objective, evaluation technique, and bibliography for the topic.
CHAPTER II

LITERAL NUMBERS AND FORMULAS

One of the first fundamental tasks which the student faces in the study of formulas and of algebra in general is to acquire a good understanding of the real meaning of literal numbers. Literal numbers can be introduced by having the student consider common situations in which the relationships between two or more elements are known. Relationships can also be stated in words and the verbal statements can then be abbreviated by substituting letters and symbols of operation and equality for the words. From this the idea is gained that a letter may stand for a meaning which can be expressed more elaborately in words.

The student must be made to understand that literal numbers represent primarily and essentially numbers.

The formula, which is considered to be the main use of algebra, provides a medium for the transition from earlier work to the more formal and systematic aspects of algebra. It is closely associated with symbolic language of literal numbers, constants, variables, substitution and evaluation, and operations with signed numbers.

The main objectives for the student in the study of literal numbers and formulas are:

1. An understanding and appreciation of the nature and significance of literal numbers and of the symbolism of algebra.
2. A clear concept of the meaning of a constant, of a variable, and of the distinction between the two.

3. The ability to set up simple formulas expressing relationships existing in situations within the student's experience.

4. Facility and accuracy in substitution and evaluation of formulas.¹

5. Pupil precision in reading and writing the symbolic language of algebra. Of particular importance is reasonable proficiency in distinguishing differences of meaning produced by the insertion or location of a symbol in an algebraic expression. For example, the pupil should be able to distinguish between $2x$ and $x^2$, between $(2x)^2$ and $2x^2$, between $2(x + 5)$ and $2x + 5$, and the like.

6. Knowledge of the usefulness of a graph or table as an aid in interpreting the formula.²

One of the chief purposes of this unit is to extend the pupil's knowledge of commonly used formulas. This implies not only familiarity with these formulas and their application, but the broader aspect of understanding the vocabulary of symbolism, translating symbols into words and specifying the order of operations.

A second purpose is to use the formula as a vehicle for introducing a variety of other important concepts and processes. That is, the new ideas will be developed as the need for them arises in connection with the interpretation of formulas. Formulas from high school science and other sources within the experiences of the boys and girls should be brought before the class as illustrations.

The use of literal numbers can be developed through numerous illustrations of the use of letters to represent such things as length of line segments, weights, sizes of angles, or unknown quantities in simple


verbal problems or equations. Such illustrations should be closely associated with repeated and supervised practice in the actual evaluation of formulas by the direct substitution of specified numbers. Performance of the indicated operations after the substitutions have been made should follow. Illustrations of these procedures can be used freely because the immature mind responds much more readily to illustration than it does to definition or verbal direction.¹

Three aspects of literal numbers that should be taught in the beginner's course of algebra are:

1. The idea should be developed that a literal number is an unknown number which is unlike the specific numbers of arithmetic. Start the discussion from a line segment of unknown length represented by a letter, such as \( l \) or \( a \). The pupil would then determine by measurement the numbers denoted by the letter \( l \) or \( a \). The student would learn how letters are used conveniently as symbols for unknown quantities whose values may be found by some process, in this case by measurement. The idea should always be made clear that literal numbers are symbols.

2. A second concept which leads to fuller understanding of the literal number is the idea of general number; that is, of a number symbol which may have any value whatsoever. This may be derived from a triangle whose unknown perimeter \( P \) is \( a + b + c \). Measurement then determines the value of the unknown number \( a + b + c \). Each new triangle will have a new value or an indefinite number of values.

3. A literal number may be a variable number which is visualized by the number scale. When a train has traveled a given number of minutes, the distance passed over may be denoted by \( d \). As the train continues to travel and the number of minutes changes, the distance represented by the literal number \( d \) also changes. The changes may be visualized by marking various values of \( d \) on the number scale. The number \( d \) is said to 'vary' with the number of minutes. It is a variable number.²

¹Butler and Wren, op. cit., p. 310.
There are also certain aspects of the formula which should be taught. These are:

1. A formula is a shorthand rule, as the rule for finding the distance traveled at a uniform rate in a given time. Likewise, formulas are used to compute interest, areas, and volumes.

2. A formula is an algebraic way of expressing relationships, such as dependence of one variable on one or more others. The Centigrade-Fahrenheit relationship is a typical example.

3. A formula may be a general solution of a given type of problem.

4. A formula is an equation. Both are solved by the same methods.

5. A formula may be represented geometrically by a graph.¹

When students learn to handle formulas the problems are solved almost automatically. The more mathematics one knows the easier life becomes, for it is a tool with which one can accomplish things that could not be done with bare hands. A thorough understanding of the working of a formula provides an assurance of being able to solve a wide variety of problems. Such an understanding includes the ability to use the formula to find one quantity, being given the values of the other quantities.

**Suggested Activities**

1. Translate statements of familiar terminology to the students into algebraic formulas. For example: distance equals rate of speed multiplied by time; cost of gasoline equals number of gallons multiplied by price per gallon.

2. Express formulas given in clear, brief algebra language into written statements: \( p = \frac{1}{2}s; A = \frac{1}{2}ab \).

3. Use a physical figure to illustrate the idea presented to make formulas more functional. For example: use the formula \( p = 4s \) for the perimeter of a square. Instead of merely substituting in the formula and finding what \( s \) is equal to, short wooden sticks can be used for illustration. Changing the length of a side and its effect on the perimeter can be shown in a more understandable manner.

4. Develop a formula to determine the amount of paint needed to paint your room. To find the number of gallons of paint needed, multiply the width of each wall by its height in feet. Add the number of feet in each wall and divide by 600. The answer will give the amount needed for one coat. To paint the outside of the house, figure the same as above but divide by 400. The answer is the number of gallons needed for two coats of outside paint.

5. Use a formula to determine the batting average of a player who was at bat 16 times and made 7 hits. Determine the batting average of your favorite baseball team.

6. John Brown earns 45 cents an hour working in a filling station. One day he wrote the formula, \( Q = \frac{1}{2}h \), to show his earnings in cents for any number of hours he worked. John then decided to make a graph to show the relation between the number of hours worked and the amount of money he earned. As the first step in making the graph John made a table, part of which is shown here:

<table>
<thead>
<tr>
<th>Amount John Earns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of hours worked</td>
</tr>
<tr>
<td>Number of cents earned</td>
</tr>
</tbody>
</table>

On a sheet of graph paper he marked a scale for hours worked and another scale for cents earned. Develop a formula and then make a table and a graph to show your earnings for two weeks.

7. Determine the number of revolutions a 28-inch bicycle wheel will make in going one mile. How many revolutions will your bicycle make in a mile?

8. In examining a catalog of athletic supplies, Ted discovered that basketballs were listed at $7.50 less 20 per cent and 10 per cent trade discounts and 2 per cent cash discount. What formula could be derived to find the actual cost of a basketball?

Search through catalogs to find other articles offered at a series of discounts. Find the net price and amount of discount on these articles so that it can be determined where to shop for the best buys.

9. The following information will help discover the formula for the circumference (perimeter) of a circle. Secure 3 or 4 circular articles such as a saucer, a metal cover, a small wheel, or the like. Measure the diameter of each. Avoid awkward fractions by using a ruler marked in inches and \( \frac{1}{10} \) of an inch. Record the measurements in a table similar to the following:
<table>
<thead>
<tr>
<th>Measurements</th>
<th>Saucer</th>
<th>Cover</th>
<th>Wheel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter (in inches)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circumference (in inches)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circumference divided by diameter</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Roll each object along a straight line and mark the length of the circumference on this line. Measure the length of the circumference in inches and 1/10 of an inch and record the measurements in the table. Next divide the circumference of each object by the diameter. Carry the division to 2 decimal places and record the answer in the table. This is the ratio of the circumference to the diameter. The circumference of a circle is always \(3.14\) times the diameter; hence, the formula for the circumference is \(c = \pi d\).

10. Find the rate of yearly interest charged for a television set bought on the installment plan. The cash price is $250 and it can be purchased for $50 down and $25 a month for 10 months. Visit local stores and bring information to class regarding the cash price and installment price for 5 articles. Use a formula to determine the rate of interest paid for installment buying on these articles. A panel discussion on the advantages and disadvantages of installment buying could conclude the lesson.

11. To determine the age of a tree measure the tree trunk's diameter in inches at chest height. Multiply by the number below which corresponds to the tree. The answer is its age. Use a formula to solve the problem.

- White elm, tulip, chestnut.............. 2
- Black walnut................................ 3
- Black oak.................................... 3
- Birch, sycamore, chestnut oak, red oak.. 4
- Ash, white ash................................ 5
- Beech, sour gum, sugar maple............. 6
- Shagbark, hickory......................... 8

12. List the sources from which money may be borrowed and the rates of interest charged in each source. The interest formula can be helpful in determining which source may be the best to use for borrowing money.

13. Using a formula in more than one way is called solving for another variable, or changing the subject of the formula, and makes the use of formulas even more convenient. From the formula \(D = RT\), find an expression for \(R\); an expression for \(T\). Select 10 or more frequently used formulas and solve each for all variables.
14. Make a list of algebraic formulas that are used in the general science class. Be aware of the use of formulas in other school subjects. Report to the class the use made of formulas in general science and other ninth grade subjects.

15. The cost of sending a telegram between 2 cities is 75 cents for the first 10 words and 4 cents for each additional word. Write a formula giving the cost of a telegram containing n words. Reduce this formula to the simplest form. Check with the local telegraph station for information on the current rates charged for sending telegrams. Prepare 5 problems with the information received using the formula previously developed.

16. Develop a formula to find the number of cubic feet in an excavation for a house which is 5 feet deep, 32 feet wide, and 45 feet long. Determine the cost of handling the dirt. Consult a local contractor to find the amount of dirt a truck can haul per load and the cost per cubic yard.

17. A cubic foot holds 4/5 of a bushel of grain or 3/5 of a bushel of potatoes. Write formulas showing how many bushels of each kind a bin will hold if its dimensions in feet are l, w, and h. Give 5 examples where formulas of this type might be used.

18. The boys in Harris school decided to try to put a new leather cover on their "mass" ball. They knew that the ball was 30 inches in diameter. How much leather would the boys need to buy to cover the surface exactly? Because the cover must be cut in strips, the boys have to allow an additional 15 per cent for waste in cutting. How much leather would they actually buy? Derive the formulas needed and bring to class 2 similar illustrations using the same formulas.

19. Develop a formula for the following mixture problem and give illustrations of similar mixture formulas that are useful. If the radiator of a car contains 10 quarts of a solution that contains 15 per cent anti-freeze compound, and 2 quarts of water are added, what per cent of anti-freeze will the radiator contain? (Radiator capacity is 12 quarts.)

20. Make a list of the formulas that have been developed in this unit.

21. The formula \( p = \frac{4s}{2} \) expresses the fact that the perimeter of a square is a function of a side.

   a) Make a table of pairs of values of the variable \( s \) and the function \( p \) in which the values of the variable have a common difference 2 and run from 0 to 10. If the values of the function have a common difference, find it.

   b) Make a table starting with \( s \) equal to 1 in which any value of the variable is double the preceding one. How is each value of the function related to the preceding one?
c) If any value of \( s \) is multiplied by 3, what relation does one find between the corresponding values of \( p \)?

d) If a side \( s \) of a square is doubled, what effect does it have on the perimeter?

**Teaching Aids**

The bulletin board is a dynamic way of using visual aids. It is the focal point for the display of material pertinent to a given unit. The general purposes for its use are to furnish aid in achieving certain objectives, to display material, to stimulate interest, and to promote an event.

The bulletin board also teaches by showing details of topics being discussed, motivates students by arousing their curiosity, develops a concept with illustrated charts and graphs, helps develop a vocabulary, serves as a basis for conversation, and can be the starting point for remedial work. Several outcomes and attitudes developed by students in preparing the bulletin board include respect for the opinions of others, the ability to work together, and skill in arranging and organizing material.

Recreational mathematic materials are being used by teachers of mathematics in secondary schools in many different ways and are often included as a part of the regular lesson plan. According to a recent survey of teachers using recreational mathematic material, most teachers felt the use of such materials improved classes and resulted in stimulating interest for mathematics in most of their pupils.

It was found that recreational mathematics was most useful in ninth and tenth grades as a part of general mathematics, elementary algebra, and plane geometry. Most of the instructors who had tried recreational
mathematics materials in the classroom were convinced of the usefulness of such materials in improving pupils' attitudes and making learning more effective. A bibliography of recreational mathematics publications is listed by Brandes in The Mathematics Teacher.¹

Blackboards and blackboard equipment, including rulers, compasses, protractors, and colored chalk are essential for the teaching of mathematics. Diagrams and illustrations should be used freely to develop concepts and to provide opportunities for the use of material learned.

Filmstrips are frequently used in the teaching of mathematics as introductory or review material. The following filmstrips and films can be purchased from the companies listed:


"Thinking in Symbols" (twenty-seven frames). McGraw-Hill Book Company, Text-Film Department, 330 West 42nd Street, New York 36, New York. Works from familiar applications back to mathematical principles. Use of symbols in mathematics is compared to use of signs and signals in everyday life.

"Grouping Symbols and Order of Operations" (thirty-four frames). McGraw-Hill Book Company, Text-Film Department, 330 West 42nd Street, New York 36, New York. Shows how use of symbols makes clear and unmistakable the proper order of operations in problem solving.

"Equations and Formulas" (sixty-three frames). The Jim Handy Organization, 2821 East Grand Boulevard, Detroit 11, Michigan. Simple equations, what they mean and how they are solved and the multiple use of formulas is shown.

"Algebra in Everyday Life" (16 mm. film-ten minutes). Coronet Films, Coronet Building, Chicago 1, Illinois. Shows how algebra is used in everyday life, as well as how it is used in specialized fields. Emphasizes the basic algebraic steps of observation, translation, manipulation, and computation.

Special Difficulties

The evaluation of formulas generally presents no learning difficulties. However, mistakes in substitution occur with unexpected frequency. These are most often associated with the rewriting or recopying of the formulas with the letters replaced by corresponding numerical values. This type of error can be corrected to a considerable degree by having the students make a practice of enclosing in a separate parenthesis each numerical value which is substituted for a letter. This helps to focus attention upon each quantity as a separate element in the formula and to avoid the confusion of one element with another.

The solution of formulas often causes students much difficulty. This is because the students are not made consciously and specifically aware of the general principles underlying the solutions. The principles are quite simple and are capable of being applied in an understanding manner by ninth grade students. These are the principles which underlie the solution of all simple equations, whether numerical or literal.¹

It is the duty of the teacher to explain fully the procedures involved in solving formulas. Each step of the solution should be carefully explained in a simple and reasonable manner.

Evaluation of Unit

Evaluation is done with certain purposes and principles in mind. Some of the purposes of evaluation are: (1) a diagnosis of pupil needs

¹Butler and Wren, op. cit., pp. 317-318.
and interests; (2) the motivation of learning activities; (3) the determination of the changes in the behavior of pupils; (4) the reporting of progress in learning to pupils and parents; (5) the appraisal of the effectiveness of materials, techniques, and methods of organization; and (6) keeping the community informed concerning the effectiveness of the school program.

The program of evaluation should: (1) appraise all of the significant outcomes included in the objectives; (2) give valid evidence of important changes taking place in pupils; (3) be an integral part of the teaching and learning process; and (4) be a continuous and cooperative process. ¹

To gain the most from this unit careful explanations by the teacher should be illustrated with numerical examples and practical applications. After explanations there should be time allowed for questions and for discussion by students and teacher. Understanding should be followed by supervised practice or drill. Diagnostic testing and remedial instruction should follow the drill.

At the completion of the unit on literal numbers and formulas the student should be thoroughly familiar with the formula as the means of representing quantitative relationships. This would include: (1) an appreciation of its application to a multitude of similar situations; (2) considerable skill in translating the symbols into words, and vice versa; (3) knowledge of the order of operations implied in an algebraic

expression; and (4) the ability to solve the formula for one variable in
terms of the others and to find the numerical value of one variable when
values of the others are given.

Bibliography

Books

Butler, Charles H., and Wren, F. Lynwood. The Teaching of Secondary

Junior High School Mathematics: Iowa Secondary School Cooperative
Curriculum Program. Department of Public Instruction,


Lieber, Lillian R. Take a Number. Lancaster: The Jaques Cattell
Press, 1946.

Mallory, Virgil S. First Algebra. Chicago: Benjamin H. Sanborn and
Company, 1950.

Multi-Sensory Aids in the Teaching of Mathematics. Eighteenth
Yearbook of the Council of Teachers of Mathematics. New York:
Bureau of Publications, Teachers College, Columbia University,
1945.

Principals, 1942.

Schaaf, William L. Mathematics for Everyday Use. New York: Barnes
and Noble, Inc., 1942.


Periodicals

Brandes, Louis Grant. "Using Recreational Mathematics Materials in
the Classroom," The Mathematics Teacher, XLVI (May, 1953), 326-329.

The Mathematics Teacher, XLI (February, 1948), 60-69.


One of the most common and important activities of algebra is the solution of linear equations. It is assumed that the student will bring to the study of algebra some understanding of what an equation means since he will have had experience with simple equations in his previous work in arithmetic. However, this experience will have been largely above the intuitive level.

These intuitive reactions are generally sufficient and satisfactory as long as the situation is very simple. The student's problem situation becomes so complex that the student is unable to keep all the elements and their numerical relationships clearly in mind at the same time introducing brackets and parentheses into equations, logic must take its place. It is then that a more exact and more formal and powerful tool for the analysis of problem situations. The algebraic equation is one such tool.

As an aid to thinking the equation breaks away the great intentions. Its function is not primarily to convey information, but rather to establish a logical relation between quantities or ideas. Once association is equated to law. The equation and its treatment is treated independently as a machine without producing such an explanation. In the equation regarding the chief purpose, in the equation, is assigned to serve merely, an aid to increase processing. Thus aid is verified as an abundance of statistical materials.
CHAPTER III

EQUATIONS

One of the most common and important activities of ninth grade algebra is the solution of linear equations. It is presumed that the student will bring to his study of algebra some understanding of what an equation means since he will have had experience with simple equations in his previous work in arithmetic. However, this experience will have been barely above the intuitive level.

These intuitive reactions are generally sufficient and satisfactory as long as the situation is very simple. The moment a problem situation becomes so involved that the student is unable to keep all the elements and their proper relationships clearly in mind at the same time intuition breaks down and, when this happens, logic must take its place. It is then that a shift must be made to more formal and powerful tools for the analysis of problem situations. The algebraic equation is a tool of this type.

As an aid to thinking the equation ranks among the great inventions. Its function is not primarily to convey information, but rather to establish a logical relation between quantities or ideas whose association is subject to law. The equation can be treated indefinitely as a machine without producing much of an impression upon the operator regarding the chief purpose which the equation is designed to serve; namely, an aid to thought processes. This fact is verified by an abundance of statistical material,
as well as by the difficulty that is commonly encountered in trying to
employ the equation in arriving at the conclusions implied by verbal
problems. All that any problem ever does is to state some facts which
challenge one's ability to determine whether those facts necessarily
predicate some conclusion.

Thus the time that it will take to drive from Detroit to St. Louis
is determined by looking up the distance on the road map, consulting one's
experience relative to speed, and then using these data to reach a con-
clusion regarding time. If the data assembled and their treatment answer
the question efficiently, it would seem to constitute a rather final indi-
cation that all of the processes which have been employed are perfect.

The ability to formulate comparisons includes a body of mechanical
forms and conventions. That is true of the equation, which is a machine
and much more, but wherever the machine is found there is a tendency to
make the wheels go around regardless of the purpose which their turning
is designed to serve. The equation is such a versatile machine that
manipulations can easily be made a varied and interesting exercise. The
equation fits into a scheme of things where its solution is incident to
a more significant situation which gives rise to the equation.

The solution of the equation requires an application of the
fundamental operations of algebra, but this is not true of the situations
which give rise to the equation. Such relations as those of time,
distance, and velocity involve ideas of dependence which are quite as
fundamental as are the operations by means of which the conclusions in
any specific instance may be reached.
The pupil should be taught to recognize relationships involved in problems and exercises because this is necessary in solving the problems. Typical examples are the relations between interest, time, and rate; the number of articles purchased and the price paid; the value of taxable property and the taxes; the value of a polynomial and the value of the variable; the cost of a railroad ticket and the distance traveled; and the postage for sending a parcel and the distance and weight.

The pupil should receive training in recognizing relationships through practice in solving equations. He should acquire the habit of analyzing the processes and operations which are to be employed in order that the value of the unknown may be determined. Emphasis on thinking will minimize the habit of thoughtless manipulation of symbols in algebra and will lead to mastery.¹

Certain objectives of equations which should be acquired in the first year of algebra include:

1. The ability to frame an equation or set of equations to express a quantitative relation when presented in problem form.
2. The ability to solve such equations or sets of equations.
3. The ability to understand the graph of a functional relation.
4. An appreciation of and familiarity with typical linear and quadratic equations.
5. The ability to solve for constants in such equations, given a pair of values of the variables.²


Timing is an essential factor in introducing equations. A preferable time for introduction is after the students have acquired a sufficient knowledge of fundamental algebraic procedures to understand what is being done in each step in the procedure. Students should be familiar with algebraic addition, subtraction, multiplication, division, and the handling of symbols before they undertake the solution of equations.

The teaching of simple equations involves six separate steps. They are as follows:

1. Intuitive period, when the child learns.
   a) Any number, except zero, divided by itself equals one.
   b) -n (any number) plus +n equals zero. This procedure is an excellent method for eliminating any term that is not desired.
   c) The 'equation idea' provides unlimited opportunities for representing relationships in mathematics, sciences, and in many everyday procedures.
   d) The appearance of an equation may be changed without relating its fundamental truth, through the operation of the four basic axioms.
   e) The equation asks a question that is answerable.

2. Question period. The child learns to ask the question indicated in a simple one operation equation, to estimate a sensible answer, and to check for accuracy.

3. How equations are built up and taken apart period. This period grows out of building identities.

4. Unknown located in each member period. This requires two steps.
   a) Simplifying each member so as to put the equation in simpler form.
   b) Changing the resulting equation to one in which the unknown occurs in only one member.

5. Accumulative equations period. This necessitates a greater knowledge of the four fundamental operations including monomials and polynomials, as well as the elementary forms of factoring.

6. Controlling equations period. Resultant of studying simple equations as a whole, understanding the common types and methods of procedure, and in devoting the greater part of the practice period to mixed groups containing all types. The child will then be faced with the necessity of thinking.¹

The student should be taught to look upon the algebraic equation as a device which enables him to investigate relationships which would be too complex to be investigated successfully or easily without its aid. It should be explained that the solution of formulas or equations operates under certain fixed laws called axioms. The axioms should be explained and illustrated by the teacher in terms of the familiar quantitative concepts of the student's past experience. After a feeling of the logic of the axioms has been given the student, practice in using them with arithmetical and literal numbers should be given.

The fundamental operational axioms involved in the solution of linear equations are as follows:

1. If equal quantities are added to equal quantities, the results are equal.

2. If equal quantities are subtracted from equal quantities, the results are equal.

3. If equal quantities are multiplied by equal quantities, the results are equal.

4. If equal quantities are divided by equal quantities, other than zero, the results are equal.¹

These axioms should become so much a part of the student that he will apply them as readily to literal numbers as to ordinary arithmetical numbers.

Linear equations assume a variety of forms in ninth grade algebra. The following examples are illustrative:

\[
ax = b \quad x + a = b \quad ax + b = c \quad ax + b = cx + d \\
\frac{x}{a} = b \quad a - x = b \quad ax + bx = c \quad x - a = b
\]

¹Butler and Wren, *op. cit.*, p. 313.
These forms are all variations of a common form, but the similarity is usually not immediately apparent to students encountering them for the first time. Emphasis is frequently deficient in this aspect. To clarify the misunderstanding which might develop, a principle can be stated in a way which is clearly understandable to ninth grade students and which is applicable to all forms of linear equations. It is formulated in the following key sentences:

1. In solving any linear equation in one unknown for the unknown (we may call it x) the object is to get an equation in which x will stand alone on one side of the equation and will not appear on the other side.

2. In order to do this we must get rid of all the other numbers or letters which are associated with x on its side of the equation.

3. We get rid of these numbers or letters by undoing the operations which associate them with x; that is, by applying the processes which are the inverse (opposite) of those which bind these letters or numbers to x.

4. If any operation is performed on one side of an equation in order to change its value, the same operation must be performed on the other side of the equation because if it is not we shall no longer have an equation.1

This principle gives a basis for the solution of linear equations without any recourse to intuition. It gives emphasis to the character of the equation and lends organization and generality to the solution of linear equations which eliminated any necessity for developing special methods for different forms. Following this general plan, the specialized procedures will emerge automatically as the student finds need for them since they are but adaptations of the general plan to particularized forms of the equation.

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1 Ibid., pp. 315-316.
Mathematical operations can be made clearer to the immature student by the simple device of asking him to answer a set of three questions: What are you doing? What for? By what right? In the solution of the equation $2x + 3 = 17$ the three aspects used are:

1. Description of the operation—subtract three from both sides of the equation.

2. The purpose of the process of subtraction is to have only the unknown and its coefficient left on one side of the equation and a number on the other side.

3. The justification of the operation or subtraction resides in the well-known law of equations.

In using this approach an attempt is made to take the mystery out of mathematical operations by making the purpose clear, to make the student conscious of the need of knowing the correct mathematical terminology to be used in describing the operation he performs, and to emphasize the fact that every operation performed in mathematics must be justified by an appropriate law of mathematics.\(^1\)

Equations may be written in more than one form but they are still equivalents. They will seem different to the pupils, and the thinking used in making these equations will differ. Pupils should be permitted to write the equations in as many ways as possible and then they should be given sufficient time in class to explain the thinking they did before writing their equations. It should be remembered that not all pupils will see a given problem situation in exactly the same way and the form of an equation that is meaningful to one pupil will not be equally so to another. Any form of an equation that is applicable to the problem situation under consideration is acceptable.

Suggested Activities

1. Determine the value of each of the following:
   a) What is the value, in cents, of 10 nickels?
   b) What is the value, in cents, of $x$ nickels?
   c) What is the value, in cents, of 10 quarters?
   d) What is the value, in cents, of $x$ quarters?
   e) What is the value, in cents, of $\$17.75$?
   f) What is the value, in cents, of $x$ nickels, $y$ quarters, and $z$ dimes?
   g) If the total value of the coins in $f$ is $\$17.75$, express the fact in the form of an equation.

2. Write equations representing the following facts:
   a) Principal = $5,000; rate of interest = $x$ per cent; interest for a year = $\$100.$
   b) Principal = $5,000; rate of interest = $x$ per cent; interest for 10 years = $\$500$.
   c) Principal = $x$; rate of interest = $y$ per cent; interest for a year = $\$x$.

3. Beside each equation below is a statement that describes the operation that one pupil thought necessary to get $x$ by itself on one side. State whether this operation is correct or incorrect. Give reasons for your answer.
   a) $x - 5 = 5$ ............... add $-5$ to each side.
   b) $x + 3 = 8$ ............... add $-3$ to each side.
   c) $\frac{x}{2} = 3$ ............... divide each side by $2$.
   d) $3x = 12$ ............... divide each side by $x$.

4. John ran a roadside refreshment stand. Fred was his helper. They agreed that John’s share of the net profits one week were $\$28.11$. If $f$ represents Fred’s share, write an equation to find the amount of money Fred should have received. Solve the equation.

5. Mr. White played 4 games of golf with scores of 95, 87, and 82 in the first 3 games. His average for the 4 games was 90. What was his score for the fourth game?
   a) Write an equation that you can use to solve the problem.
      Use $x$ to represent the unknown score.
   b) Solve the equation to find the answer.

6. One season a hockey team played 19 games. The team won 4 more than twice the number of games lost. Write an equation that you can use to find how many games were won. Use $w$ for the number of games lost. Solve the equation and give the number of games lost and won.
7. A lever will balance when the product of a weight at one end and its distance from the fulcrum is equal to the product of a weight at the other end and its distance from the fulcrum. A weight of 50 pounds is placed at one end of a lever 5 feet from the fulcrum, and a weight of 25 pounds is placed x feet from the fulcrum at the other end. Write an equation for finding the distance x. Solve the equation.

8. John weighs 75 pounds and Robert weighs 90 pounds. John must sit 6 feet from the fulcrum on a teeterboard in order to balance Robert, since the product of the weight at one end and its distance from the fulcrum must be equal to the product of the weight at the other end and its distance from the fulcrum. Write an equation for finding Robert's distance from the fulcrum. Solve the equation.

9. A car uses gasoline at a rate of 1 gallon for each 20 miles. Write an equation to represent the fact that the number of gallons of gasoline used by the car varies directly as the distance traveled. Use n to represent the number of gallons of gasoline and d to represent the number of miles of distance. Solve the equation.

10. To get to a certain town I should allow myself a certain fixed amount of time. I find that if my rate of walking is 5 miles per hour I get there 1/2 hour too late; but if my rate is 8 miles per hour, I get there 1/2 hour too early. Find the distance to the town and the required time to get there. Develop similar rate-time-distance problems from your own experience.

11. If at 3:00 P.M. I have 1/2 hour in which to catch a train but I arrive 3 minutes late, at what time do I arrive?
   a) If the distance to the train is 4 miles, what is my rate of walking in miles per hour?
   b) If the distance to the train is 4 miles, what should have been my rate of walking in order to catch the train?

12. Mr. Sanders has 96 feet of netting which he wishes to use to enclose a rabbit yard. How large a yard (length and width) can be enclosed if it is twice as long as it is wide?

13. Tom earned 90 cents an hour. He worked 56 hours one week. If Tom received "time and a half" for all work over 40 hours, how much did he earn? Write an equation to find Tom's earnings for the week and solve it.

14. The area of a rectangle is 288 square inches:
   a) What is the value of A?
   b) Construct a table of corresponding values of w and l for
      \[ w = 6, 12, 18, 24, 30, 36, 42, 48. \]
c) Use the table to find the effect of the length if starting
with \( \frac{v}{w} = 6 \), we make \( \frac{v}{w} \) 2 times, 3 times, 4 times as large,
and so on.
d) Find the dimensions if the length is 8 times the width.
e) Find the dimensions if the length equals the width.
f) Find the dimensions if the length is 12 inches greater than
the width.

15. Two brands of tomatoes are for sale in the grocery. A 1\( \frac{1}{4} \)-ounce
can of the one kind costs 22 cents, while a 13-ounce can of the
other costs 20 cents. The quality is the same. Which is the
more economical? Write an equation and solve it.

16. A farmer bought 130 pounds of twine in preparation for his wheat
harvest. He paid the dealer $15.00 and received 5 cents in change.
How much did he pay per pound for the twine?

17. Last week Mr. Brown sold sweet corn at 4 ears for 25 cents. On
his best day he took in $12.50 from his sale of sweet corn. How
many ears of corn did he sell that day?

18. Ideas for understanding equations:

23. a) \( 5b + 3 = 18 \). What number multiplied by 5 and the result
increased by 3 gives 18? Is there a number which satisfies
this relation?

b) Line segments can be used to illustrate the solutions of an
equation: \( 3a + 2 = a + 8 \).

\[
\begin{array}{c|c}
3a & 2 \\
\hline
a & 8 \\
\end{array}
\]

A study of the diagram reveals: \( 2a = 6 \)
\( \therefore a = 3 \)

c) Another method is the use of the graph. With \( d \) as an
independent variable, graph the function \( 2d + 3 \). The
value of \( d \) corresponding to the function value of 7 is
the root of the equation. This is seen to be 2.

\[ 2d + 3 = 7 \]
\[ f(d) = 2d + 3 \]
19. Write an equation which will express the relationship between \( m \) and \( n \) in the following table and complete the table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>11</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>15</td>
<td>21</td>
<td>36</td>
<td>39</td>
<td>66</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

20. John wishes to make a picture frame using 65 inches of picture moulding. In order that the frame may have the right shape, the width of the frame will be .62 of the length. How long shall each of the 4 sides of the frame be? Write the equation and solve it.

21. The Smith family is planning to buy a television set. Each week they are setting aside 8 per cent of the cost of the set, or $16. Use an equation to find out how much the set costs.

22. Bring to class a list of examples showing the use of equations in science or shop courses. This list can be started at the beginning of the unit on equations and handed in at the close of the unit. These lists could serve as material for class discussion on the use of equations in everyday life.

23. In a school cafeteria in 1 week, 3 times as many bottles of milk were sold as plates of ice cream. If the sale of milk was 1,440 bottles, how many dishes of ice cream were sold? Form an equation and solve it.

---

**Teaching Aids**

Motivation for the unit can be enriched by a bulletin board display of pictures, data, and miscellaneous information showing occupations and activities where knowledge of equations is essential. This can be a class project. Pupils should be asked to start collecting the material as soon as they begin work on the unit.

Frequent use should be made of blackboards to draw diagrams illustrating equations whose solutions require the various operations ordinarily used.

The beam balance is one of the best devices for demonstrating the basic principles of simple equations. If such a balance is not
available, the following substitute can be used. Materials needed for construction include one meter stick or yardstick, one set of weights with hooks or substitutes for these, one small box containing chalk or coins to give any convenient weight, a drill and wrapping twine, and one box of tinker toys.

Steps in the construction are: (a) drill a hole equidistant from the ends and from the sides of the meter stick and attach twine by which the balance can be hung or held; (b) drill several holes similarly spaced at each end of the meter stick from which weights may be suspended; (c) with small tinker toy sticks make a fulcrum with a pointed stick in the center spool; and (d) make the balance beam from little sticks with a spool at each end so that any combination of sticks and spools may be added.

To illustrate the subtraction axiom on the meter stick beam, balance the box and one or two small weights at one end with the necessary combination of weights on the other end. Then find the weight of the box by removing the other weights from that end and the equivalent weights from the other end.

On the tinker toy beam, balance a spool with one or two small sticks attached by the necessary number of sticks on the other end. Then show the value of the spool without the sticks by removing the small sticks from the other end.

To illustrate the addition axiom on the meter stick beam, balance the box minus a few coins or pieces of chalk with the necessary number of weights on the other end. Then complete the unknown by adding coins or chalk to the unknown and the equivalent weights on the other side.
On the tinker toy beam, balance a spool with short sticks in all but one or two holes with the necessary number of sticks on the other end. Then complete the fan by adding one or two small sticks to the spool and balancing with the equivalent weight in sticks on the other end.

Similar illustrations can be worked out for the other axioms. Individual pupil investigation after class demonstration may help some pupils to grasp and retain the idea that an equation is a statement of balance and that this relationship must be preserved.

The following filmstrips, which can be purchased from the company listed, are frequently used as introductory or review material in the teaching of equations.

"Systems of Equations" (thirty frames). McGraw-Hill Book Company, Text-Film Department, 330 West 42nd Street, New York 36, New York. By reference to specific problems at sea, filmstrip develops concepts of systems of equations: consistent, inconsistent, or dependent. Solution by substitution, elimination, or addition, subtraction, or by elimination of one unknown and solving for the other. Solution by substitution is worked out and a general method derived to be applied to similar problems.

"Equations and Formulas" (sixty-three frames). The Jam Handy Organization, 2821 East Grand Boulevard, Detroit 11, Michigan. Shows simple equations, what they mean and how they are solved.

"Introduction to Equations" (fifty-seven frames). Society for Visual Education, Inc., 1345 W. Diversey Parkway, Chicago 14, Illinois. Has original diagrams and photography from daily environment explaining basic algebraic concepts and uses. Chief criticism of transposition is that it is a mechanical process which many students use without understanding involved. However, it does.

Special Difficulties

Some of the chief errors frequently encountered in the teaching of equations include combining similar terms improperly, incomplete solution, combining dissimilar terms, errors in signs in division, and the inverted result in division.
Following suggestions offered earlier in this chapter will eliminate some of the common errors mentioned. All steps in solving the equation should be written in order to fix the process well in mind. Careful explanations should be given as the idea of the equation is developed so that students may visualize the process of solution. All numbers representing the same kind of quantities should be expressed in the same units; that is, distance can be expressed either in yards or miles in the same unit.

Learning to check all answers is a very important part of the work on equations. For every word problem that a pupil solves he should satisfy himself about two things: (1) the equation he writes must fit the facts given in the problem, and (2) the answer he gets by solving the equation must be reasonable according to the conditions of the problem. The solution of an equation for a word problem is worth little unless the meaning of the problem is clearly shown by the equation solved. For every problem pupils should check the equation they write by seeing that it fulfills the conditions of the problem and should check the solution of the equation by substituting the value found for the unknown.

The process of transposition is a controversial issue among many authors and teachers and there is no general agreement for its use. The chief criticism of transposition is that it is a mechanical process which many students use without understanding the procedure involved. However, it does save time and effort in subsequent mathematics courses and if used should not be introduced until the students have had sufficient drill in solving equations.
Effective study habits which include cultivating the power of concentration, practicing careful interpretative and critical thinking, proceeding with work systematically and thoroughly, learning accurately the fundamental concepts and processes, clarifying thoughts by writing or expressing them orally, and analyzing and checking results contribute to the success of the student in mathematics. Through the development of these habits a more thorough understanding of the subject matter is attained.

In solving equations of the first degree in one unknown, the student achieves the primary skill of performing the fundamental operations with algebraic quantities. Such mechanical operations in algebra are the necessary preparatory skills which should be learned in order to solve important problems in mathematics and related fields.

**Evaluation of the Unit**

Evaluation procedures are similar for the various units of algebra. In addition to the evaluation techniques and purposes suggested in Chapter II, other means of evaluating may include one or more of the following.

Class discussion offers the teacher an opportunity to learn the quickness of perception, the degree of insight and understanding, and the ability of the student to reason logically. In addition, the teacher is given the opportunity to note the student's ability to rationalize, interpret, and remember the results established. This gives a rough appraisal of the ability of the students to learn mathematics with an economy of time and effort. The chief aim is the development of certain mathematical concepts so that both teacher and students contribute to this end.
An oral quiz often suffices to obtain desired information concerning the knowledge of a class about a particular unit of work. Its chief advantage lies in the fact that a large amount of material can be covered in a short time. The oral quiz can be used effectively to secure sufficient information to guide the teacher in teaching new units of work and in determining whether certain concepts have been sufficiently understood.

Short written tests are useful as a teaching aid, especially during the process of assimilation of new subject matter. These are usually brief objective tests which consume only a few minutes of the class period and are given frequently during the study of a particular unit of work. A full period examination is then given covering the entire unit.¹

Inventory tests for diagnostic purposes may be used at the beginning of a unit of work to find out what items need special attention and what items need little or no direct teaching. This procedure saves valuable time that can be spent on other related topics of common interest and thus motivate the classwork.

The purpose of a diagnostic test in teaching is to analyze the difficulties of a student in a particular phase of work. The aim is to reveal accurate information concerning weaknesses in order to overcome them by concentrated action.

An achievement test given at the conclusion of the unit is used to measure the degree of mastery of skills, fundamental concepts, processes, and general knowledge of a subject which has been attained.

Two of the chief purposes governing the effective use of tests are: (1) to provide the teacher with information which will make his teaching more effective; and (2) to indicate those students who need remedial teaching, special tutoring, or transferring to other work. A good testing program helps the teacher remember which of the essential elements of the course should be carefully taught. It is of vital importance in motivating students to exert greater effort to master the instructional material. It provides a permanent record of the accomplishments of each student which is useful for future reference.  

At the completion of the unit on equations a student should feel confident of and be able to demonstrate the following abilities. The student should be able to solve equations of the first degree in one unknown and be able to explain the process and meaning of each step. He should be able to state, interpret, and use the rules for operations with equal quantities and with equations. He should be able to graph simple equations of the first and second degree in two unknowns and be able to frame equations for verbal problems of moderate difficulty.

If the results of the final examination are satisfactory, the work should proceed with the next unit. If only a few students fail to show satisfactory achievement, these may be given the necessary remedial work outside the class period while the classwork moves forward. If results show that the class as a whole has not attained sufficient understanding, it will be necessary to reteach the essential elements of the unit and give another examination.

\[\text{\textsuperscript{1}Tbid., p. 121.}\]
Bibliography

Books


Periodicals


A clear understanding of certain concepts would precede the study of the operations with directed or signed numbers. These concepts include:

- directed or signed numbers, negative numbers, positive numbers, the sign of a number, the absolute value of a number, the positive and negative of each number, the number axis, the origin, the meaning of a number, negatively directed or positively directed with respect to another number.

The objectives for the teaching of signed numbers are:

1. The student should gain an understanding of the meaning of directed numbers.

2. The student should be led to see that the operations with directed numbers are consistent with the operations of arithmetic and that they constitute a short-cut method procedure in which the operations of arithmetic enter only in special cases.

3. The student should gain speed and facility in performing the fundamental operations with directed numbers which include the ability to perform the following...
Experience has shown that it is better to teach signed numbers after the students have become thoroughly familiar with some of the basic concepts of literal numbers and formulas. The study of directed numbers is an integral part of algebra and is one of the most difficult topics to develop successfully.

A clear understanding of certain concepts should precede the study of the operations with directed or signed numbers. These concepts include: directed or signed numbers, negative numbers, positive numbers, the sign of a number, the absolute value of a number, the reading and writing of arbitrary starting point in the number scale, the directions on the scale, the meaning of a number negatively directed or positively directed with respect to another number.

The objectives for the teaching of signed numbers are:

1. The student should gain an understanding of the meaning of directed numbers.
2. The student should be led to see that the operations with directed numbers are consistent with the operations of arithmetic and that they constitute a more generalized procedure in which the operations of arithmetic appear as special cases.
3. The student should gain considerable facility in performing the fundamental operations with directed numbers which include the ability to perform the following.¹

¹Butler and Wren, op. cit., p. 326.
The fact that students often fail to attain adequate mastery of these objectives undoubtedly accounts for a great deal of the difficulty which they experience in the study of algebra.

The student's numerical experience prior to the introduction of the concept of negative numbers will have been confined entirely to dealings with the numbers of arithmetic; that is, with numbers representing quantities actually greater than zero. Previous to this time the student has used zero in two capacities, either as a number or as a placeholder in writing numbers such as 306 and 400. Now it becomes necessary to give zero a new significance. Zero will now be regarded as an arbitrary starting point in the number scale from which one may count in either direction. Numbers counted in one direction will be called positive numbers while numbers counted in the opposite direction will be called negative numbers. Illustrations of this new use should be numerous.

The number scale is probably the most satisfactory and helpful of all devices for making clear the nature of positive and negative numbers and for illustrating their characteristics of oppositeness, direction, and position. It should be used in connection with other devices for illustrating the opposite character of positive and negative numbers and the arbitrary selection of the zero or reference point. Illustrations of assets and liabilities, north and south latitudes, and temperatures above

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1 Schaaf, op. cit., p. 318.
and below zero are helpful in developing the fundamental concepts.
However, they fail to make clear that whether a number is to be regarded
as a positive, negative, or zero relative to some other number, depends
not only upon its own position in the scale but also upon the position of
the number to which it is referred, and that number may or may not be the
previously established zero.

Illustrations that can be used to motivate the teaching of signed
numbers are shown in Figure 1 below. To show positive and negative numbers
which are in daily use by the public, the teacher needs only to refer to
the financial pages of the daily paper or to the daily weather reports.
These concrete illustrations should clearly establish what constitutes a
negative number, the zero point, and the positive number. By emphasizing
these interpretations, algebra becomes a vivid subject, rich in real
meaning.

<table>
<thead>
<tr>
<th>Negative Number</th>
<th>Zero Point</th>
<th>Positive Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 degrees below zero</td>
<td>Zero</td>
<td>70 degrees fahrenheit</td>
</tr>
<tr>
<td>Pupil's home, 10 houses</td>
<td>School building</td>
<td>Pupil's home, 2 houses east</td>
</tr>
<tr>
<td>west of school</td>
<td>Today</td>
<td>Day after tomorrow</td>
</tr>
<tr>
<td>Five days ago</td>
<td>Normal weight</td>
<td>Overweight 3 pounds</td>
</tr>
<tr>
<td>Underweight 8 pounds</td>
<td>No money and no debts</td>
<td>Cash on hand, $10.00</td>
</tr>
<tr>
<td>A debt of $12.00</td>
<td>Purchase price</td>
<td>Sale at gain of $9.00</td>
</tr>
<tr>
<td>Sale at loss of $50.00</td>
<td>Surface of lake</td>
<td>Height of hill 80 feet above lake</td>
</tr>
<tr>
<td>Depth of water in lake</td>
<td></td>
<td>surface</td>
</tr>
<tr>
<td>40 feet</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1.--Examples of negative number, zero point, positive number.
To fully understand the use of the four fundamental processes with signed numbers, the following concrete illustrations will help to motivate the unit, promote understandings, and make it easier for the pupils to acquire the skills in computation with directed numbers.

1. Addition:

<table>
<thead>
<tr>
<th>Original Value</th>
<th>Change in Value</th>
<th>New Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) A man has $350.00</td>
<td>He incurs a debt of $475.00</td>
<td>Find balance $(+350) + (-475) = -125$</td>
</tr>
<tr>
<td>b) Elevator on 11th floor</td>
<td>Ascends 5 floors</td>
<td>What floor does it reach? $(+11) + (+5) = +16$</td>
</tr>
<tr>
<td>c) Construction began 3 years ago</td>
<td>Estimated time for completion of contract, 8 years</td>
<td>When should work be done? $(-3) + (+8) = +5$</td>
</tr>
<tr>
<td>d) Score in card game 12 in the hole</td>
<td>Player gets set 45 points</td>
<td>What is new score? $(-12) + (-45) = -57$</td>
</tr>
</tbody>
</table>

2. Subtraction:

<table>
<thead>
<tr>
<th>Subtrahend</th>
<th>Minuend</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Airplane is up 1,250 feet</td>
<td>A few minutes later height is 700 feet</td>
<td>How much did it go down? $(+700) - (+1250) = -550$</td>
</tr>
<tr>
<td>b) Phil has won 15 points in a game</td>
<td>James has lost 12 points</td>
<td>How many points is James behind Phil? $(-12) - (+15) = -27$</td>
</tr>
<tr>
<td>c) Joe descends 400 feet from a mountain camp</td>
<td>Dick descends 120 feet from the same camp</td>
<td>How far below Joe is Dick? $(-300) - (-400) = +100$</td>
</tr>
<tr>
<td>d) Road sign west of Buffalo reads Buffalo 40 miles</td>
<td>Road sign west of Buffalo reads Buffalo 10 miles</td>
<td>How far did Mr. Evans drive toward Buffalo? $(-10) - (-40) = +30$</td>
</tr>
</tbody>
</table>

3. Multiplication:

a) A man went to work for $6.00 a day. This is a positive number because it is what he is earning. If he has worked 4 days, the question may be asked: "How much has he earned since he began work?" Time since an activity began is positive. The solution is $(+4)(+6) = +24$. 
b) Suppose a man is out of work. The cost of his room and board

today is $2.00 a day. Of course this is an expense to him and is

written negative. If he has been out of work 10 days, he may wish to

know how much he has fallen behind financially since his

unemployment began. As before the time is positive. The

solution is \((+10)(-2) = -20\). The minus sign in the answer

c) represents a decrease in his cash.

Another question might be asked about this same man. How much

c) better off was he 10 days ago? Now time ago is negative.

d) Hence the solution is \((-10)(-2) = +20\). The result is positive

e) because he had more money at the time he lost his job than he

has now.

4. Division:

a) Mr. Peterson in his will gives $5,000.00 to be divided equally

among his 5 children. How much does he give to each?

\((-5000)/(+5) = -1000\).

b) An aviator left an airport 600 miles behind him 5 hours ago.

Find his average speed. Distance backward is negative; time

ago is negative. The solution is \((-600)/(-5) = +120\). The positive

answer corresponds to the fact that he is flying ahead.

c) How long will it take to sink a shaft in a mine to a depth of

300 feet at the average rate of 15 feet a day? Both numbers

are negative and the result, which is positive, shows that

the time when the shaft will be finished will be in the future.

**Suggested Activities**

1. The figures below show the earnings of a store for 9 different

years. A plus sign at the left of a number indicates a profit,

while a minus sign at the left indicates a loss. In this case

zero, which represents no profit or loss, is the starting point.

During what years did the store have a profit? During what years

did it have a loss? What does the zero for 1936 tell?

<table>
<thead>
<tr>
<th>Year</th>
<th>Earnings (Dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1920</td>
<td>$20,000</td>
</tr>
<tr>
<td>1924</td>
<td>$40,000</td>
</tr>
<tr>
<td>1928</td>
<td>-$10,000</td>
</tr>
<tr>
<td>1932</td>
<td>-$6,000</td>
</tr>
<tr>
<td>1936</td>
<td>0</td>
</tr>
<tr>
<td>1940</td>
<td>$12,000</td>
</tr>
<tr>
<td>1944</td>
<td>$45,000</td>
</tr>
<tr>
<td>1948</td>
<td>$62,000</td>
</tr>
<tr>
<td>1952</td>
<td>$77,000</td>
</tr>
</tbody>
</table>
2. Four boys had the following scores in a game: John, 50 points; Tom, 28 points; Henry, 11 points; and Art, 20 points in the hole. Write each boy's score, using a plus or minus sign. Explain why you use the plus or minus sign in each case.

3. The pairs of quantities given below have opposite meanings. Write each quantity with a plus or a minus sign.
   a) A gain of $500.00; a loss of $150.00.
   b) A $200.00 increase; a $155.00 decrease.
   c) A fall of 15 degrees in temperature; a rise of 30 degrees.
   d) A 20 yard loss in football; a 10 yard gain.
   e) One hundred feet below sea level; 980 feet above sea level.
   f) List 10 examples similar to these above illustrating positive and negative numbers.

4. Record the temperature taken each hour for 12 hours on a graph. The following example can be used as an illustration:

```
\begin{center}
\begin{tikzpicture}
    \begin{axis}[
        width=\textwidth,
        height=0.5\textwidth,
        axis x line=bottom,
        axis y line=left,
        xlabel={Time in hours},
        ylabel={Temperature (°F)},
        xtick={6,7,8,9,10,11,12},
        ytick={-6,-4,-2,0,2,4,6},
        xticklabel style={/pgf/number format/1000 sep={,}},
        yticklabel style={/pgf/number format/1000 sep={,}},
        xticklabels={6,7,8,9,10,11,12},
        yticklabels={-6,-4,-2,0,2,4,6},
    ]
    \addplot+[mark=none,thick,red]coordinates{
        (6,2) (7,4) (8,6) (9,4) (10,2) (11,0) (12,-2)
    };
\end{axis}
\end{tikzpicture}
\end{center}
```

a) How are the above zero temperatures indicated on the graph?
b) How are the below zero temperatures indicated?

5. The figure below shows the proper position for the two scales on Monday morning. There are no profits and no losses to show, so the zeros, or starting points, are directly above one another. By Monday evening Mr. Jones had made a profit of $4.00. How do you know that for the day this means a change of $4? Look in the positive direction to find $4 on Scale A. Next find the number on Scale B that is directly below $4 on Scale A. The number on Scale B is also $4.

--- Scale A: Daily Change Scale

<table>
<thead>
<tr>
<th>---</th>
<th>Negative</th>
<th>Positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9</td>
<td>-8</td>
<td>-7</td>
</tr>
<tr>
<td>-6</td>
<td>-5</td>
<td>-4</td>
</tr>
<tr>
<td>-3</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

--- Scale B: Net Profits or Losses Sale

---
a) On Tuesday Mr. Jones began with a net profit of $4.00; so Scale A is placed with the 0, or starting point, over \( \frac{1}{4} \) on Scale B. For Tuesday he had a profit of $3.00. Is \( \frac{1}{4} \) in the positive or negative direction from 0 on the daily change scale? It is directly above what number on the net profits or losses scale?

b) On Wednesday Mr. Jones lost $2.00. What is the sum of \(-2\) added to the \(\frac{1}{4}\) on Scale B?

c) Make a tracing on thin paper of the two scales shown above. Cut them apart carefully so you can move the change scale. Use the scales to do a series of problems adding positive and negative numbers.

6. Using a similar scale to the one described in No. 5, let Scale A be a Temperature Change Scale and Scale B a Temperature Scale. A series of hourly temperatures can be used to illustrate the process of subtraction of signed numbers. Practice using the scale with a series of readings to establish the rule used for subtracting signed numbers.

7. The process of multiplication is hard to show by the use of number scales. Instead it will depend upon the student’s ability to think about numbers and to understand what happens when multiplying with signed numbers. Think of money earned as a positive number and of money spent as a negative number. The present is the starting point, or 0. Time in the past will be represented by negative numbers and time in the future by positive numbers.

a) Mr. Harris knows he is going to earn $5.00 per day for 4 days. He will have $20.00 more in 4 days than he has now. Is it correct to show the money earned per day as \( \frac{5}{4} \)? How do you know it is correct to show the number of days as \( \frac{1}{4} \)? How do you know the sign of the answer is correct? \( \frac{5}{4} \left( \frac{1}{4} \right) = \frac{5}{20} \).

b) Miss Smith knows she is going to spend $5.00 per day for 4 days. She will have $20.00 less in 4 days than she has now. How do you know that it is correct to show money spent as \(-5\)? Is the sign of the number of days correct? How do you know that the answer is correct? \( -5 \left( \frac{1}{4} \right) = -20 \).

c) Jim has earned $5.00 per day for the past 4 days. He had $20.00 less 4 days ago than he has now. Why is the amount per day written as a positive number? Why is the number of days written as a negative number? How do you know that the answer should be a negative number? \( \frac{5}{4} \left( -\frac{1}{4} \right) = -20 \).
Helen has been spending $5.00 per day for 4 days. She had
$20.00 more 4 days ago than she has now. Why are both the
amount per day and the number of days written as negative
numbers? How do you know that the answer should be a
positive number? \((-5) \cdot (-4) = 20\).

Work a series of problems of signed numbers similar to the
exercises above and develop rules for multiplying with signed
numbers.

Division with signed numbers is the opposite of the process of
multiplication. Using illustrations similar to those in No. 7
develop exercises to illustrate the rules of division of signed
numbers.

Make a wall chart of temperatures. Get a piece of wrapping paper
11 feet long. Make a bar on it 10 feet, 6 inches long. Start
the lower end of the scale at -273 degrees centigrade. Make a
mark every 10 degrees (1 inch apart) up to 1000 degrees centigrade.
Place the centigrade reading above the bar and the fahrenheit
reading below the bar. Put interesting temperatures at proper
points along the scale. Include the boiling point of water,
the freezing point of water, and the temperature of the human body.

To save time and work in finding the average of several large
numbers, the following method based on signed numbers can be used.

Choose a number between the lowest value and the highest
value given.

Find how much each value differs from your guess. Express
the difference (deviation) as a signed number.

To find the average deviation, add the deviations and divide
them by the number of cases.

Add the assumed average and the average deviation.

Proper use of the materials and illustrations mentioned previously
will motivate the students to describe other situations in which it is
convenient to indicate direction by using positive and negative numbers.

Displays should be made of available tables, charts, and graphs that show
the use of signed numbers. A real thermometer will help the slower pupils.
Boys who have access to tools and materials may make large number scales out of plywood to be used for demonstration purposes in the class. These scales should not be labeled, but the numbers should be marked in different colors so they can be easily distinguished.

3. Construction of a scale for use in developing the concept of signed numbers is a helpful aid. Materials needed for the scale include one strip of stiff cardboard five inches by twenty inches, two dozen spring type clothespins, red and black ink and pen or crayons. On the upper edge of the cardboard strip starting a quarter inch from the lefthand end, mark off twenty-six three-quarter inch spaces. Number these division points consecutively from minus twelve to plus fourteen. Set the numbers back one-half inch from the edge and use red ink for negative numbers and black ink for positive values. Color one clothespin black or some other convenient color.

This scale may be used to develop the rules for each of the fundamental processes by working out a sufficient number of examples on it so that the pupils begin to recognize the pattern and suggest rules for themselves. If the teacher has some other favorite method of introducing the rule of signs, this may be used for additional testing or for illustrating the rules. It may also be used to represent abstract number combinations; or to illustrate common situations such as temperature change, profit and loss, or points won or lost in a game.

The manipulation of the scale for each process is described below:

1. Addition.—In all cases the starting point for the first addend is zero, and the values to the right of zero or changes in value from left to right are considered positive, and values to the left of zero or changes in values from right to left are considered negative.
2. Subtraction.--The difference between two algebraic quantities is usually defined to be the number of units that must be combined with the subtrahend to produce the minuend together with the sign indicating the direction of this operation. Since the starting point for the difference is the subtrahend and differs in each case, mark that point with a black or colored pin which will not be considered as part of the result.

3. Multiplication.--Multiplication by a positive multiplier is usually considered to mean the repeated addition of the multiplicand the number of times indicated by the multiplier, starting from zero. Multiplication by a negative multiplier may be considered as "negative additions" or repeated subtractions of the multiplicand the number of times indicated by the multiplier, starting from zero.

4. Division.--Since multiplication represents repeated additions, division may then be considered as a system for repeated subtraction of the divisor from the dividend. The quotient is the number of subtractions necessary to reduce the dividend to zero or a remainder less than the divisor. Similarly, a negative quotient represents the number of "negative subtractions" or additions needed to reduce the value of the dividend to zero or a remainder.

Filmstrips can also be used as aids in teaching signed numbers.

The following may be purchased from the company listed:


"Addition and Subtraction of Signed Numbers" (forty-eight frames). Same company as above.

"Multiplication and Division of Signed Numbers" (forty-nine frames). Same company as above. Each filmstrip black and white, captioned.

"Positive and Negative Numbers" (sixty frames). The Jam Handy Organization, 2821 Grand Boulevard, Detroit 11, Michigan. The "well-known positive" and the "helpful negative" visualized and how they influence each other in the four fundamental operations. The realm of the negative and its contribution to mathematical scope is also illustrated.

Special Difficulties

Difficulties usually encountered in pupils in the teaching of signed numbers are computation, understanding and applying principles, unfamiliar terms, and personal and mental traits.
The largest number of errors usually occur in subtraction and are caused by one or more of the following reasons: limited span of attention, abrupt transition from simple to complex processes, or failure to give properly balanced attention.

Errors resulting from understanding and applying principles are primarily due to the pupil's inexperience with the major principle of the unit in life situations as attributed to the illustrations used in the text and the methods used by the teacher. Often there is lack of knowledge concerning the principles underlying the rules. Due to lack of training the students are unable to apply the principles to life situations.

Unfamiliar terms are used frequently and cause many errors. The student's mathematical vocabulary is limited and the use of foreign terms in directions results in misunderstanding. Rather than define a term, students will guess or follow a wrong cue.

Personal and mental traits usually involve carelessness, initiative, intelligence, and cooperativeness. A lack of interest in mathematical relationships can be a cause of errors or careless manipulations. Several solutions can be used to eliminate many of the common errors mentioned. Solutions for many of the difficulties are described below.

The unit in signed numbers should be introduced inductively, which involves the development of a reasonable comprehension of negative numbers. This comprehension is effectively developed by the pupil's analysis and interpretation of familiar experiences that constitute effective illustrations of the principles.

The principle should be developed in terms of procedures used by persons who have frequent occasion to apply the principle to concrete
situations in real life; for example, the clerk's method of subtracting
when making change, the lineman's method of calling "downs," and the
methods used in quantity are indicated with reference to some zero point.

2. The essential skills involved in the unit should be reduced to
negative numbers are involved as coefficients.
3. The ability to add, subtract, multiply, and divide
the elements and the elements should be taught one at a time until they
are mastered. Each operation should be practiced separately with examples
of the same type before the student is expected to apply both operations
to the same example.

5. The pupil should be taught to make specific applications of the
principle. This requires practice in relating the principle to a variety
of situations belonging to the same type, such as the measurement of
thermometer readings or altitude charts.

The technical terms that are essential to the pupil's comprehension
and his effective use of the principle should be distinguished from those
that are not essential but which are equally hard to understand. Easy
synonyms should be substituted for the technical terms that are not
essential.

The personal traits found to be directly related to success in the
given unit should be systematically developed at any and all appropriate
times. These should not be left to the pupil's own initiative.¹

Evaluation of the Unit

The evaluation procedures and techniques previously developed for
Chapters II and III also apply for the unit on signed numbers. Reference
to these techniques can be made by reviewing the chapters mentioned.

¹Douglas Waples and Charles A. Stone, The Teaching Unit, pp. 186-


Periodicals


BIBLIOGRAPHY

Note: The literature used in compiling the needed information for each unit developed in elementary algebra is listed at the end of each chapter. The materials which were used to determine the characteristics of a resource unit are listed here.

Books


Articles


